

E_{11} and Supersymmetry

Duncan Steele and Peter West
Department of Mathematics
King's College, London WC2R 2LS, UK

We introduce fermions into the E_{11} non-linear realisation. We show, at low levels, that the commutators of the Cartan involution invariant subalgebra of E_{11} with the known supersymmetry transformations of eleven dimensional supergravity lead to symmetries of the theory indicating the consistency of supersymmetry and E_{11} .

1. Introduction

The first paper which conjectured E_{11} symmetry [1] only considered the bosonic sectors of the theories under consideration and the same is true for all subsequent papers. However, there have been a number of results which follow from E_{11} symmetry which have traditionally been, or have subsequently been, shown to follow from supersymmetry. Two such examples are the two and five form central charges in the l_1 representation of E_{11} , which is conjectured to contain all brane charges, [2] and the representations carried by form fields which imply the classification of gauged supergravities [3,4]. A brief account of some of the evidence for an underlying E_{11} symmetry of strings and branes is summarised in the first seven pages of [5].

The dimensionally reduced maximal supergravities contain non-linear realisations that encode the scalar fields. In fact this statement is true for all supergravity theories which possess scalars in their supergravity multiplet. The prototype example is the maximal supergravity theory in four dimensions which possesses an E_7 non-linear realisation with local subgroup $SU(8)$ [6]. The other fields in the supergravity multiplet transform as matter representations of the non-linear realisation, that is under the local subgroup. This includes the fermions.

The local subalgebra adopted in the non-linear realisation of E_{11} is the Cartan involution invariant subalgebra denoted $K(E_{11})$. The commutation relations of this latter algebra were given at low levels in [2] and it was found that the generators could be represented at low levels by the eleven dimensional γ -matrices. As such $K(E_{11})$ possesses, at low levels, a 32 component spinor representation that might be used as the supersymmetry parameter [2].

It has also been proposed [7] that non-linearly realised E_{10} is a symmetry of maximal eleven dimensional supergravity. This is a subalgebra of E_{11} , but it differs from the earlier proposal [1] in the way it incorporates space-time; the fields are taken to depend only on time and the spatial derivatives of the fields are proposed to occur at higher level in E_{10} . Fermions have been incorporated in the E_{10} non-linear realisation [8-11]. Following the pattern found in supergravity theories in lower dimensions these authors took the fermions to belong to linear representations of $K(E_{10})$. They found that there exists at low levels a representation which is a vector spinor of the ten dimensional Lorentz group which can be identified with the gravitino and that this is a representation at all levels, albeit an unfaithful one. The previously found [2] thirty two component unfaithful representation, which is a spinor of the ten dimensional Lorentz group, was used as the supersymmetry parameter.

In this paper we follow a similar path to incorporate the fermions into the E_{11} non-linear realisation. The contents of this paper are as follows. In section two we summarise the algebra of $K(E_{11})$, in section three we compute the $K(E_{11})$ transformations of the fields at low levels, in section four we will find a unfaithful representation of $K(E_{11})$ which can be identified with the gravitino, in section five we compute the commutators of the low level fields between the known supersymmetry transformations and their previously found $K(E_{11})$ transformations and show that they are consistent in that they lead to known symmetries of the theory.

As very briefly indicated in reference [11] some calculations incorporating fermions,

which are unpublished, have been carried out in the E_{11} context by these authors.

2. The Cartan Involution invariant subgroup of E_{11}

In this section we summarise the commutation relations of E_{11} generators and those of the Cartan involution invariant subgroup of E_{11} , denoted $K(E_{11})$. The Dynkin diagram of E_{11} is given by

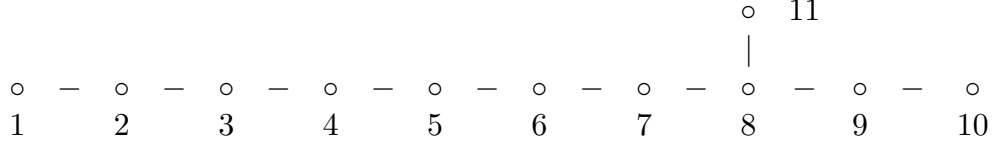


Figure 1. The Dynkin diagram of E_{11}

Deleting node eleven we find the algebra $GL(11)$, which corresponds in the non-linear realisation to eleven dimensional gravity. As such it is natural to decompose the adjoint representation of E_{11} in terms of $GL(11)$ which consists of $SL(11)$ and the remaining generator of the Cartan subalgebra. We denoted these generators by K^a_b , $a, b = 1, \dots, 11$ and they obey the commutators

$$[K^a_b, K^c_d] = \delta^c_b K^a_d - \delta^a_d K^c_b \quad (2.1)$$

All generators of a Kac-Moody algebra are formed from multiple commutators of the Chevalley generators. The level of a generator is defined to be the number of times the Chevalley generator E_{11} (not to be mistaken with the symbol for the algebra itself) occurs for positive root generators, or minus the number of times F_{11} appears for negative root generators. The results of the decomposition can be classified by this level [12,7]. The positive root generators at level one and two respectively are given by[1]

$$R^{a_1 \dots a_3}, R^{a_1 \dots a_6} \quad (2.2)$$

while

$$R_{a_1 \dots a_3}, R_{a_1 \dots a_6} \quad (2.3)$$

are the negative root generators at levels -1 and -2 respectively.

The commutation relations of the positive root generators with $GL(11)$ are

$$[K^b_c, R^{a_1 \dots a_3}] = 3\delta^{[a_1}_c R^{b|a_2 a_3]}, \quad [K^b_c, R^{a_1 \dots a_6}] = 6\delta^{[a_1}_c R^{b|a_2 \dots a_6]} \quad (2.4)$$

While the commutators of the negative root generator with those of $GL(11)$ are given by

$$[K^b_c, R_{a_1 a_2 a_3}] = -3\delta^b_{[a_1} R_{c|a_2 a_3]}, \quad [K^b_c, R_{a_1 \dots a_6}] = -6\delta^b_{[a_1} R_{c|a_2 \dots a_6]} \quad (2.5)$$

Generators at level two, or minus two, can be found as the commutator of two level one, or minus one, generators.

$$[R^{a_1 \dots a_3}, R^{a_4 \dots a_6}] = 2R^{a_1 \dots a_6}, \quad [R_{a_1 \dots a_3}, R_{a_4 \dots a_6}] = 2R_{a_1 \dots a_6} \quad (2.6)$$

In these equations we have chosen the normalisation of these generators.

Finally, the commutators between the positive and negative root generators at levels one and two are [1]

$$\begin{aligned}
[R^{a_1 \dots a_3}, R_{b_1 \dots b_3}] &= 18\delta_{[b_1 b_2}^{[a_1 a_2} K^{a_3]}_{b_3]} - 2\delta_{b_1 \dots b_3}^{a_1 \dots a_3} (\sum_b K^b_b) \\
[R^{a_1 \dots a_3}, R_{b_1 \dots b_6}] &= \frac{5!}{2} \delta_{[b_1 \dots b_3}^{a_1 \dots a_3} R_{b_4 \dots b_6]} \\
[R^{a_1 \dots a_6}, R_{b_1 \dots b_6}] &= -5! \left(9\delta_{[b_1 \dots b_5}^{[a_1 \dots a_5} K^{a_6]}_{b_6]} - \delta_{[b_1 \dots b_6]}^{[a_1 \dots a_6]} \sum_c K^c_c \right) \\
[R_{b_1 \dots b_3}, R^{a_1 \dots a_6}] &= \frac{5!}{2} \delta_{b_1 \dots b_3}^{[a_1 \dots a_3} R^{a_4 \dots a_6]}
\end{aligned} \tag{2.6}$$

The usually adopted Cartan involution of a Lie algebra is defined on the Chevalley generators as

$$E_a \rightarrow -F_a, F_a \rightarrow -E_a, H_a \rightarrow -H_a \tag{2.7}$$

The effect on the generators used above is

$$K^a_b \rightarrow -K^b_a, R^{a_1 \dots a_3} \rightarrow -R_{a_1 \dots a_3}, R^{a_1 \dots a_6} \rightarrow R_{a_1 \dots a_6} \tag{2.8}$$

The Cartan involution invariant subalgebra $K(E_{11})$ is generated by the invariant combination of the Chevalley generators given by

$$S_a = E_a - F_a \tag{2.9}$$

A basis for the Cartan involution invariant subalgebra $K(E_{11})$ is given, up to and including level 2, by [2]

$$\begin{aligned}
J^{ab} &= K^a_c \eta^{cb} - K^b_c \eta^{ca} \\
S_{a_1 \dots a_3} &= R^{b_1 \dots b_3} \eta_{b_1 a_1} \dots \eta_{b_3 a_3} - R_{a_1 \dots a_3} \\
S_{a_1 \dots a_6} &= R^{b_1 \dots b_6} \eta_{b_1 a_1} \dots \eta_{b_6 a_6} + R_{a_1 \dots a_6}
\end{aligned} \tag{2.10}$$

The J^{ab} s generate the Lorentz algebra. In fact, these generators are not the generators which are invariant under the Cartan involution of equation (2.7), but under the modified Cartan involution given by $E_a \rightarrow -\eta_{aa} F_a$, $F_a \rightarrow -\eta_{aa} E_a$ and $H_a \rightarrow -H_a$. This introduces the Minkowski metric η_{ab} , which ensures that we have the Lorentz group $SO(1,10)$ rather than the group $SO(11)$. We could also have worked with the Cartan involution of equation (2.7) but then Wick rotated to Minkowski signature at any stage.

The commutators between the generators of $K(E_{11})$ are given by [2]

$$\begin{aligned}
[J^{ab}, J^{cd}] &= -\eta^{bd} J^{ac} - \eta^{ac} J^{bd} + \eta^{bc} J^{ad} + \eta^{ad} J^{bc} \\
[S^{a_1 \dots a_3}, S_{b_1 \dots b_3}] &= 2S^{a_1 \dots a_3}_{a_4 \dots a_6} - 18\delta_{[b_1 b_2}^{[a_1 a_2} J^{a_3]}_{b_3]} \\
[S_{a_1 \dots a_3}, S^{b_1 \dots b_6}] &= -3S^{b_1 \dots b_6}_{[a_1 a_2, a_3]} - \frac{5!}{2} \delta_{[a_1 \dots a_3]}^{[b_1 \dots b_3} S^{b_4 \dots b_6]}
\end{aligned} \tag{2.11}$$

The first generator on the right-hand side of the last equation is the Chevalley invariant combination of the level three and minus three generators that we include for completeness, although it is beyond the level truncation used in this paper.

3. The action of $K(E_{11})$ on the bosonic fields

In this section we calculate the transformations of the bosonic fields under rigid $K(E_{11})$. By definition, the group element from which the nonlinear realisation is constructed transforms under rigid transformations as

$$g \rightarrow g_0 g, \quad g \rightarrow g h \quad (3.1)$$

where $g_0 \in E_{11}$ is a rigid, i.e. constant, transformation, but $h \in K(E_{11})$ is a local transformation.

Using the analogue of the Iwasawa decomposition we may write the general group element of E_{11} as

$$g = e^{h^a H_a} e^{\sum_{\alpha} A^{\alpha} E_{\alpha}} e^{\sum_{\beta} B^{\beta} S_{\beta}} \quad (3.2)$$

Where the sums over α and β run over all positive roots, and S_{β} denotes an element of $K(E_{11})$. This group element is of the form of an element of the Borel subalgebra multiplied by a Cartan involution invariant group element. Using the local symmetry we can choose the group element to be of the form

$$g = e^{h_a{}^b K^a{}_b} e^{A_{a_1 \dots a_3} R^{a_1 \dots a_3}} e^{A_{a_1 \dots a_6} R^{a_1 \dots a_6}} \dots \quad (3.3)$$

Thus we choose our coset representatives. We note that we did not use the local Lorentz group part of $K(E_{11})$ to choose the $h_a{}^b$ to be symmetric.

Carrying out a rigid Borel transformation takes us from one coset representative to another, so we can immediately read off the transformation of the fields. However, this is not the case for a general E_{11} transformation and one has to perform an additional compensating local $K(E_{11})$ transformation to bring the group element back to being one of the coset representatives.

We must also include in the group element a part associated with space-time, that is a factor $e^{x^a P_a}$. In principal we should add further generators associated to the generalised space-time introduced in [2] corresponding to the non-linear realisation of $E_{11} \otimes_s l_1$, but these are likely to lead to higher order effects than those being considered in this paper. As such we take all E_{11} generators except those of $GL(11)$ to commute with P_a .

We now consider the rigid transformation

$$g_0 = e^{c^{a_1 \dots a_3} S_{a_1 \dots a_3}} \quad (3.4)$$

All other $K(E_{11})$ transformations can be found from this one by taking commutators. As a result we find at lowest order in the transformation parameter

$$\begin{aligned} \delta_{c^3} e_{\mu}{}^a &= 2c^{\nu\rho\lambda} (9A_{\mu\nu\rho} e_{\lambda}{}^a - A_{\nu\rho\lambda} e_{\mu}{}^a) \\ \delta_{c^3} A_{a_1 \dots a_3} &= -3A_{a_2 a_3 b} (9c^{\mu\nu\rho} e_{\mu}{}^c e_{\nu}{}^d e_{\rho}{}^b A_{cda_1} - c^{\mu\nu\rho} e_{\mu}{}^c e_{\nu}{}^d e_{\rho}{}^e A_{cde} \delta_{a_1}^b) \\ &\quad + 60A_{a_1 \dots a_3 cde} c^{\mu\nu\rho} e_{\mu}{}^c e_{\nu}{}^d e_{\rho}{}^e + c_{\mu\nu\rho} e_{a_1}{}^{\mu} e_{a_2}{}^{\nu} e_{a_3}{}^{\rho} - c^{\mu\nu\rho} e_{\mu a_1} e_{\nu a_2} e_{\rho a_3} \end{aligned} \quad (3.5)$$

The object $e_{\mu}{}^a$ is the vielbein and how it enters into the non-linear realisation is discussed in appendix A. The quantity $c^3 \equiv c^{a_1 a_2 a_3}$ is the same constant regardless of

whether it carries flat or curved indices. In other words we do not use the vielbein to convert the flat indices to the curved indices on c^3 , but rather show explicitly the vielbein factors that are present. In other words, $c^{\mu\nu\rho} = \delta_a^\mu \delta_b^\nu \delta_c^\rho c^{abc}$ is also a constant.

To find the above result one must first move the g_0 of equation (3.4) past the $e^{h\cdot k}$ factor in the group element g of equation (3.3) using the equation

$$\begin{aligned} g_0 e^{h\cdot k} &= e^{h\cdot k} e^{-h\cdot k} e^{c^{a_1 a_2 a_3} S_{a_1 a_2 a_3}} e^{h\cdot k} \\ &= e^{h\cdot k} \exp\{c_{\mu_1 \mu_2 \mu_3} e_{a_1}^{\mu_1} e_{a_2}^{\mu_2} e_{a_3}^{\mu_3} R^{a_1 a_2 a_3} - c^{\mu_1 \mu_2 \mu_3} e_{\mu_1}^{a_1} e_{\mu_2}^{a_2} e_{\mu_3}^{a_3} R_{a_1 a_2 a_3}\} \end{aligned} \quad (3.6)$$

The presence of the vielbeins in equation (3.6) is explained in the appendix. Moving the expression in equation (3.6) after the $e^{h\cdot k}$ factor past the next factor in the group element g , namely $e^{A_{a_1 a_2 a_3} R^{a_1 a_2 a_3}}$ the $e^{c^{\mu_1 \dots \mu_3} e_{\mu_1}^{a_1} e_{\mu_2}^{a_2} e_{\mu_3}^{a_3} R_{a_1 \dots a_3}}$ term creates a GL(11) transformation that must be reordered in the group element. Similar considerations apply to the passage of $e^{c^{\mu_1 \dots \mu_3} e_{\mu_1}^{a_1} e_{\mu_2}^{a_2} e_{\mu_3}^{a_3} R_{a_1 \dots a_3}}$ past the factor containing the six form field. Finally, one can recognise $e^{c^{\mu_1 \dots \mu_3} e_{\mu_1}^{a_1} e_{\mu_2}^{a_2} e_{\mu_3}^{a_3} R_{a_1 \dots a_3}}$ as part of the compensating local transformation

$$h = e^{c^{\mu\nu\rho} e_\mu^a e_\nu^b e_\rho^c S_{abc}} \quad (3.7)$$

We note that this contains a term $e^{c^{\mu\nu\rho} e_{\mu a} e_{\nu b} e_{\rho c} R^{abc}}$ which must be reabsorbed into the change in the three form field together with the similar term that arises from the passage of the factor $e^{c_{\mu_1 \mu_2 \mu_3} e_{a_1}^{\mu_1} e_{a_2}^{\mu_2} e_{a_3}^{\mu_3} R^{a_1 a_2 a_3}}$ in equation (3.6).

To calculate the variation of the vielbein under S_6 we repeat this procedure with $g_0 = e^{c_{a_1 \dots a_6} S^{a_1 \dots a_6}}$ and a suitably chosen compensating local transformation. We find the result

$$\delta_{c^6} e_\mu^a = 5! c^{\nu_1 \dots \nu_6} A_{\nu_1 \dots \nu_6} e_\mu^a - 5! 9 c^{\nu_1 \dots \nu_6} A_{\nu_1 \dots \nu_5 \mu} e_{\nu_6}^a - \frac{5! 9}{2} c^{\nu_1 \dots \nu_6} A_{\nu_1 \dots \nu_3} A_{\nu_4 \nu_5 \mu} e_{\nu_6}^a \quad (3.8)$$

Finally we write down the effect of a rigid Lorentz transformation on the vielbein in this formalism so as to fix the normalisation. That is we take $g_0 = e^{c_{ab} J^{ab}}$ and process it as in equation (3.6) to find a local transformation. The result is

$$\delta_{c^2} e_\mu^a = 2 c_\mu^\nu e_\nu^a \quad (3.9)$$

where the second index on c_μ^ν is simply now written as an upper index.

4. Spinorial representations of $K(E_{11})$

In this paper we wish to include fermions in the E_{11} non-linear realisation. As we have already mentioned the prototypical example is the maximal supergravity in four dimensions which has an E_7 symmetry [6]. In this theory, and indeed all supergravity theories in which the scalars are part of the supergravity multiplet, the spinors appear in the nonlinear realisation as matter representations. The matter representations transform as a linear representation of the chosen local subalgebra, which is $SU(8)$ in the example just considered. This is the Cartan involution invariant subalgebra and so the maximal compact subgroup of E_7 . We note that once one has chosen a coset representative, one must in general carry out compensating local transformations, which act on matter representations.

Spinors have already been introduced in the E_{10} approach [8-10] where they also took the spinors to transform under the Cartan involution invariant subalgebra. To construct the representation of $K(E_{10})$ appropriate to the gravitino, these authors started with the vector spinor representation of $SO(10)$ and introduced a transformation for S_3 , up to level three, that satisfied the known commutation relations for the $K(E_{10})$. It turned out that it was enough at low levels to introduce only the gravitino field and so the representation found was highly unfaithful.

These techniques also apply to E_{11} and we also take the gravitino to be a matter representation. We start with the standard Lorentz transformation of the gravitino $SO(10,1)$ with a tangent space vector index;

$$J_{ab}\psi_c = -\frac{1}{2}\gamma_{ab}\psi_c - 2\eta_{c[a}\psi_{b]} \quad (4.1)$$

To find a suitable transformation of the vector spinor under S_3 we write down all possible terms with the correct $SO(1,10)$ character and demand that it obey the algebra given in equation (2.11) involving the S_3 generator. In particular from the commutator between two S^3 generators in equation (2.11), one derives the following two relations

$$[S^{abc}, S_{ade}] = 0, \quad [S^{abc}, S_{abd}] = -J^c_d \quad (4.2)$$

Where a, b, c, d, e are distinct indices. The second relation relates the S_3 transformation back to the known $SO(1,10)$ transformation of equation (4.1). Given the S_3 transformation we can find all higher level $K(E_{11})$ transformations by taking repeated commutators. We find that the transformations of the vector spinor, that is the gravitino, up to level two, are given by

$$\begin{aligned} S_{abc}\psi_d &= \frac{1}{2}\gamma_{abc}\psi_d - \gamma_{d[ab}\psi_{c]} + 4\eta_{d[a}\gamma_b\psi_{c]} \\ S_{abcdef}\psi_g &= -\frac{1}{4}\gamma_{abcdef}\psi_g - 2\gamma_{g[abcde}\psi_{f]} + 5\eta_{g[a}\gamma_{bcde}\psi_{f]} \end{aligned} \quad (4.3)$$

One can repeat this procedure, starting with the spin 1/2 representation of $SO(10,1)$ and recover the result [2]

$$J_{ab}\psi = -\frac{1}{2}\gamma_{ab}\psi \quad S_{abc}\psi = \frac{1}{2}\gamma_{abc}\psi \quad S_{abcdef}\psi = -\frac{1}{4}\gamma_{abcdef}\psi \quad (4.4)$$

5. Commutator of $K(E_{11})$ and Supersymmetry

In this section we will calculate the commutator of the supersymmetry variations and the $K(E_{11})$ transformations on the vielbein and the three form. For our supersymmetry variations we take the well known transformations from eleven dimensional supergravity. We will find that the commutators result in symmetries of the theory and so demonstrate the consistency of E_{11} with supersymmetry at least at low levels. This is far from guaranteed as E_{11} has so far been based entirely on the bosonic fields. We take the supersymmetry transformations of the vielbein, the three form, and its dual, the six form with the Grassmann parameter ϵ_α to be [13]

$$\begin{aligned} \delta_\epsilon e_\mu^a &= \bar{\epsilon}\gamma^a\psi_\mu \\ \delta_\epsilon A_{\mu\nu\rho} &= \frac{1}{2}\bar{\epsilon}\gamma_{[\mu\nu}\psi_{\rho]} \\ \delta_\epsilon A_{\mu_1\dots\mu_6} &= -\frac{1}{60}\bar{\epsilon}\gamma_{[\mu_1\dots\mu_5}\psi_{\mu_6]} + \frac{1}{2}\bar{\epsilon}\gamma_{[\mu_1\mu_2}\psi_{\mu_3}A_{\mu_4\dots\mu_6]} \end{aligned} \quad (5.1)$$

We note that the normalisation of the fields was already determined by their appearance in the E_{11} group element of equation (3.3) and those chosen in equation (5.1) are the ones compatible with this previous choice.

One finds that the commutator of the variation of Q_α and S_3 on the vielbein is given by

$$[\bar{\epsilon}Q, c^3 \cdot S_3]e_\mu{}^a = \frac{1}{2}c^{bcd}\bar{\epsilon}\gamma_{bcd}\gamma^a\psi_\mu - 4c_\mu{}^{ad}\bar{\epsilon}\psi_d - 4c_\mu{}^{cd}\bar{\epsilon}\gamma^a{}_c\psi_d + 4c^{acd}\bar{\epsilon}\gamma_{\mu c}\psi_d + c^{bcd}\bar{\epsilon}\gamma^a{}_{\mu bc}\psi_d \quad (5.2)$$

When carrying out this calculation it is important to remember that the $K(E_{11})$ transformation of the gravitino discussed in section three was defined in the tangent frame, however the gravitino in the supersymmetry transformations has a curved index, so when considering the $K(E_{11})$ variation of the gravitino, we must include the vielbein required to convert a flat to a curved index, that is $\psi_\mu = e_\mu{}^a\psi_a$. The same applies to the threeform which we must write as $A_{\mu\nu\rho} = e_\mu{}^ae_\nu{}^be_\rho{}^cA_{abc}$.

From equation (5.2) we extract the generic form of the commutator

$$[Q, S_{bcd}] = \frac{1}{2}\gamma_{abc}Q + \left(\frac{1}{2}\gamma^f{}_{ebc} + 2\eta_{eb}\delta_c^f - 2(\eta_{eb}\gamma^f{}_c - \delta_b^f\gamma_{ec})\right)\psi_d J_L{}^e{}_f \quad (5.3)$$

which we recognise as a supersymmetry transformation and a **local** Lorentz transformation denoted by the symbol $J_L{}^e{}_f$. On the metric, which is a Lorentz invariant object, the field dependent Lorentz transformations do not appear, and we are left with $[Q, S_{abc}] = \frac{1}{2}\gamma_{abc}Q$. This is expected, because the supercharge Q is a spinor, which transforms as in equation (4.4).

We note that the commutator (5.3) is field dependent. This is a well known phenomenon that occurs in the commutator of supersymmetry and gauge transformations when some of the fields have been set to zero using the supermultiplet of gauge symmetries, the prototype example is to fix the Wess-Zumino gauge in supersymmetric Yang-Mills theory; for a review see [14]. It is to be expected here as we have used a local symmetry, that is the $K(E_{11})$, to gauge away the non-Borel part of the group element.

A similar calculation on the threeform field gives

$$[\bar{\epsilon}Q, c^3 S_3]A_{\mu\nu\rho} = \frac{1}{4}c^{abc}\bar{\epsilon}\gamma_{abc}\gamma_{\mu\nu}\psi_\rho \quad (5.4)$$

The spacetime threeform is a Lorentz invariant object, and one does not expect to see the field dependent terms of equation (5.3). Thus one finds that the generic commutator of a supersymmetry transformation with a rigid $S^{a_1 a_2 a_3}$ transformation is, up to level two, of the form

$$[Q, S_{abc}] = \frac{1}{2}\gamma_{abc}Q \quad (5.5)$$

plus local transformations.

The commutator of supersymmetry and S_6 on the vielbein is given by

$$\begin{aligned} [\bar{\epsilon}Q, c^6 S_6]e_\mu{}^a = & + \frac{1}{4}c^{\nu_1 \dots \nu_6}\bar{\epsilon}\gamma_{\nu_1 \dots \nu_6}\gamma^a\psi_\mu \\ & - \frac{5!}{2}c^{\nu_1 \dots \nu_6}\bar{\epsilon}\gamma_{\nu_1 \nu_2}\psi_{\nu_3}(9A_{\mu\nu_4\nu_5}e_{\nu_6}{}^a - A_{\nu_4 \dots \nu_6}e_\mu{}^a) \\ & + 2c^{\nu_1 \dots \nu_6}\bar{\epsilon}\gamma^a{}_{\mu\nu_1 \dots \nu_5}\psi_{\nu_6} - 20c_\mu{}^{a\nu_1 \dots \nu_4}\bar{\epsilon}\gamma_{\nu_1 \dots \nu_3}\psi_{\nu_4} \\ & - 5(c_\mu{}^{\nu_1 \dots \nu_5}\bar{\epsilon}\gamma^a{}_{\nu_1 \dots \nu_4}\psi_{\nu_5} - c^{a\nu_1 \dots \nu_5}\bar{\epsilon}\gamma_{\mu\nu_1 \dots \nu_4}\psi_{\nu_5}) \end{aligned} \quad (5.6)$$

These variations lead to the commutator relation

$$[Q, S_{a\dots f}] = \frac{1}{4}\gamma_{a\dots f}Q - 30\gamma_{ab}\psi_c S_{def} + \left(\frac{5}{2}(\delta_{fj}\gamma^k{}_{a\dots d}\psi_e - \delta_f^k\gamma_{ja\dots d}\psi_e) + \gamma^k{}_{ja\dots e}\psi_f - 10\delta_{ej}\delta_f^b\gamma_{a\dots c}\psi_d\right)\psi_f J_L^j{}_k \quad (5.7)$$

where the right hand side is understood to be antisymmetrised over the indices $abcdef$. Thus we may write the commutator as

$$[Q, S_{abcdef}] = \frac{1}{4}\gamma_{abcdef}Q \quad (5.8)$$

plus local transformations. We note that equations (5.5) and (5.8) are compatible with regarding the supercharge as a spinor which we found to transform as in equation (4.4). The commutators of the Cartan involution subalgebra with the supersymmetry as anticipated in [2].

Appendix A. Vielbeins in E_{11}

In the calculations given in this paper the vielbein plays an important role, and in this appendix we briefly discuss how the vielbein appears in the E_{11} non-linear realisation. For this purpose we can take our group element to contain just the part appropriate for gravity, namely

$$g = e^{x^a P_a} e^{h_a{}^b K^a{}_b} \dots \quad (A.1)$$

where \dots indicates factors involving higher level fields. The most direct way to see the presence of the vielbein is to compute the Cartan form

$$\mathcal{V} = g^{-1}\partial_\mu g = dx^\mu e_\mu{}^a (P_a + (e^{-1}\partial_\mu e)_a{}^b K^a{}_b + \dots) \quad (A.2)$$

where $e_\mu{}^a = (e^h)_\mu{}^a$. The Cartan forms transform under the local subalgebra $K(E_{11})$ as $\mathcal{V} \rightarrow h^{-1}\mathcal{V}h + h^{-1}dh$. At lowest level this is just the Lorentz group and so $e_\mu{}^a$ transforms on its upper a index just like a vector under the Lorentz group while any reparameterisation, more precisely any $GL(11)$ transformation, of x^μ gives a corresponding change in the lower μ index of $e_\mu{}^a$. Thus $e_\mu{}^a$ does transform as a vielbein should. Indeed constructing the theory of gravity from the non-linear realisation as was first done in [15], and again in a more vielbein orientated approach in [16], one finds that $e_\mu{}^a$ does indeed appear in the theory as the vielbein should.

Effectively, the above calculation of the vielbein evaluates $e^{h_a{}^b K^a{}_b}$ in the vector representation as this factor acts on P_a . In this representation

$$(K^a{}_b)_c{}^d = \delta_c^a \delta_b^d \quad (A.3)$$

where c, d are the representation matrix indices clearly giving $(e^{h_a{}^b K^a{}_b})_c{}^d = e_c^{hd}$ in vector representation.

In the paper we encounter expressions where we move $e^{h_a{}^b K^a{}_b}$ past generators in representations of $GL(11)$, for example equation (2.4). In particular we find that

$$e^{-h\cdot K} c^{a_1\dots a_3} R_{a_1\dots a_3} e^{h\cdot K} = c^{\mu\nu\rho} e_\mu{}^a e_\nu{}^b e_\rho{}^c R_{abc} \quad (A.4)$$

We recall that the parameter c^3 is the same constant no matter what indices it displays, but it is natural to write its indices so as to reflect what it is contracted with. We also give the analogous result for the positive root generators

$$e^{-h \cdot K} c_{a_1 \dots a_3} R^{a_1 \dots a_3} e^{h \cdot K} = c_{\mu\nu\rho} e_a^\mu e_b^\nu e_c^\rho R^{abc} \quad (A.5)$$

which involves the inverse vielbeins $e_a^\mu = (e^{-h \cdot K})_a^\mu$.

8. References

- [1] P. West, *E₁₁ and M theory*, Class. Quant. Grav. **18** (2001) 4443, [hep-th/9501068](#)
- [2] P. West, *E₁₁, SL(32) and Central Charges* Phys. Lett. B **575** (2003) 333, [hep-th/0307098](#).
- [3] F. Riccioni, Duncan Steele, Peter West, *E₁₁ origin of all gauged supergravities*; JHEP **0707** (2007) 63, [arXiv:0705.0752](#).
- [4] E. A. Bergschoeff, T. Nutma and I. De Baetselier, *E₁₁ and the embedding tensor*, JHEP **9** (2007) 047, [arXiv:0705.1304](#)
- [5] F. Riccioni and P. West, *E(11)-extended spacetime and gauged supergravities*, JHEP **0802** (2008) 039, [arXiv:0712.1795](#)
- [6] E. Cremmer and B. Julia, *The SO(8) supergravity*, Nucl. Phys. B **159** (1979) 141
- [7] T. Damour, M. Henneaux and H. Nicolai *E₁₀ and a Small Tension Expansion of M Theory* Phys. Rev. Lett. **89** (2002) 221601 [hep-th/0207267](#)
- [8] S. de Buyl, M. Henneaux and L. Paulot, *Extended E₈ Invariance of 11-Dimensional Supergravity* JHEP **0602** (2006) 056 [hep-th/0512292](#)
- [9] T. Damour, A. Kleinschmidt and H. Nicolai *Hidden symmetries and the fermionic sector of eleven-dimensional supergravity* Phys. Lett. B **634** (2006) 319 [hep-th/0512163](#)
- [10] S. de Buyl, M. Henneaux and L. Paulot *Hidden Symmetries and Dirac Fermions* Class. Quant. Grav. **22** (2005) 3595 [hep-th/0506009](#)
- [11] M Henneaux, E Jamsin, A Kleinschmidt and D Persson; Phys. Rev. D (2009) 045008; [arXiv:0811.4358](#)
- [12] M. R Gaberdiel, D. I. Olive and P. West; *class of Lorentzian Kac-Moody algebras* Nucl. Phys. B **645** (2002) 403 [hep-th/0205068](#)
- [13] E. Cremmer, B. Julia, and J. Scherk, *Supergravity Theory In Eleven Dimensions*, Phys. Lett. B **76** (1978) 409-412; Igor Bandos, Nathan Berkovits and Dmitri Sorokin, *Duality-Symmetric Eleven-Dimensional Supergravity and its Coupling to M-Branes*, Nuclear Physics B **522** (1997) 214-233, [hep-th/9711055](#).
- [14] P. West, *Introduction to Supersymmetry and Supergravity*, World Scientific (1990)
- [15] A. Borisov and V. Ogievetski, *Theory of dynamical affine and conformal symmetries as the theory of the gravitational field*, Teor. Mat. Fiz. **21** (1974) 329
- [16] P. West, *Hidden Superconformal Symmetry in M Theory*, JHEP **08** (2000) 007, [hep-th/005270](#)